First semestral exam - November 17, 2023
B. Math. (Hons.) 2nd year
Group Theory
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Each question carries 9 points.

Q 1. Let H, K be subgroups of a group G. Prove that, for any $a, b \in G$, the sets HaK and HbK are either disjoint sets or the same set. Further, show that

$$|HaK| = |H \cap aKa^{-1}|\ |K|.$$

OR

If H is a subgroup, and N is a normal subgroup of a finite group G such that $O(H)$ and $O(G/N)$ are relatively prime, then prove that $H \leq N$.
Hint. Consider the left action of H on G/N.

Q 2. If a p-group P acts on a set S whose size is coprime to p, show that G must fix a point. If P is a p-Sylow subgroup of a group G, prove that its action by left multiplication on the set $G/N_G(P)$ of left cosets of $N_G(P)$ has a unique fixed point which is the identity coset.
Hint. How many p-Sylow subgroups does $N_G(P)$ have?

OR

If H, K are subgroups of a group G, denote by $[H, K]$, the subgroup generated by elements of the form $hkh^{-1}k^{-1}$ with $h \in H, k \in K$. If $D_0(G) := G, D_{i+1}(G) := [D_i(G), D_i(G)]$, show that every automorphism $\theta : G \to G$ maps each $D_i(G)$ into itself.

Q 3. Let G_n be the group of invertible $n \times n$ matrices with entries from a field K such that $g_{ii} = 1$ for all i and $g_{ij} = 0$ for all $i > j$. Let $n \geq 3$.
(i) Find the center of G_n.
(ii) If $K = \mathbb{F}_p$, for a prime p, prove that G_n is a p-Sylow subgroup of $GL_n(\mathbb{F}_p)$.
Hint. The order of $GL_n(\mathbb{F}_p)$ is $(p^n - 1)(p^n - p) \cdots (p^n - p^{n-1})$.

OR

(i) Prove that $\mathbb{Z}_3$ is not isomorphic to a quotient group of S_4.
(ii) If p is a prime, and P is a subgroup of S_p, of order p, then $|N_{S_p}(P)| = p(p-1)$.
Hint for (i). If it is, what is the order of the kernel?
Hint for (ii). A subgroup of order p in S_p is cyclic generated by a p-cycle. Also, for any finite group G and a subgroup H, the number of different subgroups of G conjugate to H equals the index of the normalizer of H in G.

Q 4. If G is a group of order 26985, prove that its center has order at least 257.
Hint. 257 is a prime number.

OR

Consider the subgroup

$$A := \{(30x + 42y, 105x + 231y) : x, y \in \mathbb{Z}\}$$

of $\mathbb{Z} \times \mathbb{Z}$. Find the smallest positive integer d such that $(d, 0), (0, d) \in A$.
Hint. The question asks for the exponent of the finite, abelian group $(\mathbb{Z} \times \mathbb{Z})/A$.

Q 5. Consider the action of $GL_2(\mathbb{R})$ on the set S of real, symmetric 2×2 matrices by $(g, A) \mapsto gAg^t$ for $g \in GL_2(\mathbb{R}), A \in S$. Describe the orbits of this action.

OR

Let G be the group of real matrices $\begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix}$ with $x > 0$.
(i) Describe the conjugacy classes in G.
(ii) Determine the set of 2×2 matrices A such that e^{tA} belongs to G for all $t \in \mathbb{R}$.